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20

RELATIVE OPTIMAL REINFORCEMENT PATTERNS FOR FIBER REINFORCED COMPOSITE MEMBRANES

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FOREWORD

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RELATIVELY OPTIMAL REINFORCEMENT PATTERNS FOR FIBER-REINFORCED COMPOSITE MEMBRANES

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1. Summary

A method is proposed to obtain relatively optimal reinforcements in a fiber-reinforced composite membrane. The method is based on the fact that:

a. A reinforcing fiber is most efficiently utilized when it coincides with the direction of maximum required stiffness.

b. The directions of principal trajectories are not dependent on any fiber reinforcement as long as the fibers coincide with those directions.

An optimality condition for the fiber reinforcement is derived. The derivation is an adaptation of one given in the literature for isotropic homogeneous materials. The optimality condition derived states that the specific strain energy divided by the specific number of fibers is a constant, for maximum stiffness at a given weight, and minimum weight for given stiffness. Since optimization of the reinforcement only is considered and optimization of the matrix is not included, the resulting optimum is relative.

Mathematically, the optimum is weak. As an example, some numerical values for a boron-epoxy wedge in tension were computed.

* Research Assistant
** Assistant Professor

2. Contents: Page

3. Notation:

(Notation used in appendix, not included)

(7)	equation (7)
[7]	Reference number [7]
a_k	number of fibers (§8.2.); $k = 1,2.$
b_1	C_E
b_2	$2C_p$
C_E	constant (§8.2.)
C_p	constant (§8.2.)
C_{ij}	stiffness coefficient; $i, j = 1,2,6.$
C_1	optimality constant (§8.2.)
dS	surface area element
e_1	ϵ_1^2
e_2	ϵ_2^2
e_3	$\epsilon_1 \epsilon_2$
n_i	exponent of a_k (§8.2.); $i = 1,2,3.$

P_i	external load; $i = 1,2,$ _ _ _
r_i	radius; figure 1; $i = 1,2.$
S_{ij}	compliance coefficient; $i, j = 1,2,6.$
U	specific strain energy (§8.2.)
U_i	component of U, dependent on a_i (§8.2.)
V	volume
W_B	work done by external loads
α	wedge angle; figure 1.
δ	variation
δ_i	displacement of loadpoint of P_i
ϵ_i	strain; $i = 1,2,6.$
σ_i	stress; $i = 1,2,6.$

4. Nomenclature:

FLAG: Fiber LAying Gadget.

FRC: Fiber Reinforced Composite

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5. Introduction

Fiber-reinforced composites (FRC's) can be of great advantage in structures or structural parts requiring a material with distinct directional characteristics. Pressure vessels, for instance, are such structures, and fiber-wound pressure vessels are examples of successful utilization of FRC's.

The directional load or stiffness requirements are not always constant throughout the structure; quite often they change rapidly, i.e. the directions and values vary considerably over a short distance. A well-known example is a strip with a hole under tension.

FRC's are most commonly applied in the form of cloth or tape made of continuous straight fibers. Therefore, in order to follow changes in directions, a laminate has to be made consisting of several layers with different orientations. It is obvious that the material is not used very efficiently in these cases of rapidly changing directions, the more so since generally the directional loads and/or stiffness requirements vary simultaneously and significantly as well. Hence it appears to be worthwhile to look into the possibility of a more satisfactory method to deal with this kind of problems.

Simply stated, the problem is how to use as little material as possible to provide as high a stiffness as possible in a given direction, and clearly this is an optimization

problem.

In this report an account is given of initial work done on a method to optimize the reinforcement. It is based on two observations:

a. For a unidirectionally reinforced FRC, the direction of maximum stiffness coincides with the fiber direction.

b. The principal stress (and strain) directions in an isotropic body do not change when the isotropic body is made specially orthotropic; i.e. when the axes of orthotropy coincide with the principal stress (strain) directions.

By laying fibers along principal trajectories determined for the isotropic case, reinforcements are obtained which are (relatively) optimal under certain conditions for the stress and/or strain fields. This is the essence of the method discussed in Section 8 of this report.

Optimum designs are often impracticable because they are too complicated and hence too expensive to fabricate. This is not likely to happen to FRC designs optimized as outlined in Section 8, for the following reasons:

a. The price of a reinforcing material like boron or graphite is so high [35] that it contributes substantially to the cost of the structure. Any weight reduction causes a similar cost reduction of the reinforcing material and hence also an overall cost reduction for the structure.

b. One of the main factors contributing to material cost is the precision required for fiber alignment. High precision is also required when the material is applied to

the structure. For the method discussed in Section 8, the material is used in the form of a single continuous wire. Hence the need for precision during fabrication has been eliminated, and of the two precision requirements, only the one for application remains. Not only will this result in lower cost but it will be beneficial also for the overall precision finally achieved.

c. The percentage of material waste will be very low for production versions of the fiber-laying device described in Section 9, since the length of the fiber needed can be computed accurately. (Production versions of the FLAG will be equipped with a fiber cutting device. Therefore, no fiber will be laid along return loops.)

In order to put the method in a proper perspective, some optimization theories and techniques used in the field of FRC's as well as a few other fields are briefly surveyed.

Finally, a description is given of an experimental Fiber-LAying Gadget (FLAG) which is being developed to lay a single continuous fiber along a curved trajectory in a flat plane. This device will eventually be capable of laying fibers in predetermined patterns, and then test specimens can be made to check theoretical results with experimental data.

6. Structural Application Methods for FRC's

In this section a brief survey is given of methods used to obtain optimal structural applications of composites reinforced with continuous fibers. Of these methods, drawbacks are indicated which have been aleviated by the method discussed in Section 8. Extensive bibliographies covering each of these methods can be found elsewhere, see for

instance Reference 1.

6.1. Lamination

Continuous fibers are often worked into tapes or woven fabrics. In a laminate made of layers of tapes and/or fabrics, the number and the orientation of the layers are variables which allow the structural designer to more or less match the material properties and the structural requirements.

Methods have been developed to determine rationally the number and orientation of the layers ([2, 3, 4] and [5], Div. 6. Propulsion Systems). Apart from the fact that sometimes these methods reportedly yield questionable results, they determine the variables in one single point only. In applications where the stress and/or strain gradient and direction do not vary considerably throughout the structure, these methods will provide useful information. However, in structures with stress concentrations they become impractical since they require application of the material in quantities and forms that are impracticable. The designer therefore is forced to compensate by using more material and thus decreasing its efficiency.

As a result, the superior directional properties are not used to full advantage in many laminated structures.

6.2. Filament Winding

Fiber-wound structures account for some of the most successful applications of FRC's. This is mainly due to the uniformity of the stress-and-strain fields in structures particularly suited for fiber winding, e.g. pressure vessels.

Fiber-winding techniques, to a certain extent, allow for aligning fibers with principal directions and controlling the number of fibers in a given direction. In some cases geometrical disturbances like openings in vessels can be incorporated in the design in such a manner that the design is optimal [1].

Stress analysis of fiber-wound structures is based either on netting analysis or on a continuum-mechanics approach [6]. In netting analysis, most frequently used in the early days of filament winding, the fibers are assumed to be the load-carrying constituents of the composite, with the matrix not playing a load-carrying role at all. Netting analysis simplifies the stress analysis of fiber-wound structures considerably and in many papers its application has been investigated, e.g. [7-11]. Studies of some special structures, not being fiber-wound structures, have also been based on netting analysis [12].

When based on continuum mechanics, the analysis of fiber-wound structures becomes complicated [6]. Since the structures under consideration have one or more axes of rotatory symmetry and also because the stress and strain fields are uniform, the complicated analyses are still tractable.

6.3. Reinforced Concrete

Reinforced concrete can be classified as a fiber-reinforced composite. The functions of reinforcement and matrix, as assumed in analyses of reinforced concrete structures, differ from those in "real" FRC's; the reinforcement is assumed to carry tensile loads while the matrix carries compressive loads.

The reason for paying attention to the field of reinforced concrete is the fact that in that field, too, there is a need for optimization of the reinforcement. And some

interesting methods have been developed, both theoretical [13] and experimental [14]. In [14] a discussion has been given on the determination of the reinforcement pattern in a reinforced concrete plate of irregular form by employing Ligtenberg's reflection-moire method. With such an experimental method, the method discussed in Section 8 could still be used in cases where analytical and numerical analyses are impractical.

7. Optimum Design

There is a distinct difference between "structural analysis" and "structural design." Analysis is the study of the behavior, i.e. stresses, deflections, etc. of a structure of given form, dimensions, and material constants. Design, or direct design as it sometimes is called, is concerned with structures in which certain parameters, e.g. plate thickness, are to be determined. These parameters are determined such that the structure not only satisfies conditions of equilibrium and compatibility but also meets one or more additional requirements. For example, the requirement that the stiffness be maximum for a given weight or the weight be minimum for a given stiffness. The structure resulting from such a structural design procedure is an optimum (or optimal) design.

On the subject of optimal structural design, a most extensive literature now exists and quite a few survey papers cover parts of the subject. Barnett's paper [15] is a good introduction (and a very readable one at that) to some optimum design topics related to the method discussed in Section 8. Also, a survey article [16] is mainly devoted to reinforced concrete optimization.

The following paragraphs in this section refer to fields in optimum design theory which are related to the problem of optimization of FRC's.

7.1. Plastic Design

Plastic design deals with structures in which plastic deformation is allowed to occur. Much progress has been made in developing the theory and many important results have been established concerning optimality conditions.

In [17], minimum weight plastic design has been related to uniform strength elastic design, based on the assumption that the yield function is directly proportional to the specific strain energy. One of the important theorems in the theory of plastic design states that a design is a minimum weight design if the rate of dissipation per unit volume is constant. Analog to this, for an elastic design to be an optimum design the specific strain energy is constant [17].

7.2. Elastic Design

Derivations of optimality conditions for elastic structures are often based on the principle of minimum potential energy. This principle states (see for instance [18], page 171) that the displacement which satisfies the differential equations of equilibrium, as well as the conditions at the bounding surface, yields a smaller value for the potential energy of deformation than any other displacement, which satisfies the same conditions at the bounding surface.

Usually, only one type of structure is treated at a time, e.g. sandwich beam, column, etc., and a linear relationship between stiffness and weight is assumed [19]. Furthermore, in most cases only one design parameter, like the thickness of a plate, is independently variable.

A generally applicable optimality condition for maximum stiffness designs has been derived by Huang [20]. His very simple derivation, also based on the principle of minimum potential energy, leads to a condition for optimality. This derivation is used in Section 8 to establish an optimality condition for the reinforcement in FRC membranes.

Since an optimization problem can be formulated as an isoperimetric problem [21, 22], variational calculus is frequently used in derivations of optimality conditions. Once an optimality condition has been established, for instance to obtain maximum stiffness for a given weight, another optimality problem with the same condition can easily be formulated to obtain minimum weight for a given stiffness. This is due to what is called the reciprocity of isoperimetric problems [21]; in the theory of mathematical programming it is known as duality (see for instance [23]). This approach has been followed in [24].

7.3. Maxwell-Michell Structures

A theory of optimum frame structures based on work by Maxwell [25], has been developed by Michell [26]. According to this theory, the bars of an optimum frame coincide with the principal strain trajectories of a displacement field involving constant principal strains the value of which is determined by means of Hooke's Law by the allowable stress and the E-modulus of the bar material. The principal strain lines can be shown to be given by the same equations as the slip lines in plane flow of a perfectly plastic solid [27, 28]. For the story of the revival of the interest in Michell structures, see [15, 27].

In [29] it has been suggested that this theory can be applied to FRC's. A laminate of varying number of layers has to be used to approximate the change in fiber

diameter required by the theory. No justification has been given for this application of the theory.

Michell structures are statically determinate and can therefore be dimensioned such that the designs are uniform strength designs and optimum designs as well. Generally, FRC structures, e.g. plates under a biaxial state of stress, are statically indeterminate and consequently applicability of the theory to FRC structures is far from being self-evident.

7.4. Systematic Structural Synthesis

The methods of optimization mentioned in the three preceding paragraphs are analytical. However, much work of a numerical nature has also been done. By means of numerical methods, cases can be investigated which are too complicated to be treated analytically, e.g. highly indeterminate structures.

Systematic structural synthesis is the procedure leading to an optimum design by iteration of an analysis-redesign cycle. For the iteration, quite a few numerical methods are known [1]. The primary requirement to be met by such methods is a rapid convergence to an absolute optimum.

Structural synthesis is basically the same trial and error design procedure as has always been used by structural designers. It differs in that the procedure has been rationalized in the sense that in the iteration process every next step yields a design that is better than or at least as good as the previous one. This is achieved by employing mathematical methods developed in the theory of mathematical programming [23, 30]. Because

of the enormous amount of numerical operations involved, these methods are practicable only if computers are used. Even then, particularly when non-linear programming methods are applied, only a limited number of design variables can be handled to prevent the problem from becoming too complicated to formulate [31].

In [32], systematic synthesis has been discussed and illustrated by an example problem. An important result reported in the paper is the observation that the minimum weight optimum design for a statically indeterminate structure is not necessarily a fully stressed design.

The significance of statical determinacy or statical indeterminacy and fully stressed design in structural optimization was not very clear in the past [15], but has since been investigated thoroughly [16, 31]: A statically indeterminate structure is a relatively optimal design, which is not fully stressed and not stiffer for the same weight nor lighter for the same stiffness than a corresponding statically determinate optimum design.

Sometimes a statically indeterminate structure can become fully stressed if it has been prestressed [33]. This fully stressed design is lighter than the optimum design obtained without prestressing.

If the domain of feasible solutions is not convex or the mathematical programming techniques do not, in general, yield absolutely optimal solutions, then other methods have to be used [34].

7.5. Conclusions from the Literature Survey

From the literature survey, some conclusions can be drawn which indicate possible directions of investigation of the optimization of reinforcement in FRC's.

Statically indeterminate optimal designs are, in general, relatively optimal in the sense that an equivalent but statically determinate design can be found which is absolutely optimal. An orthogonally stiffened plate in some instances can be considered as a statically determinate structure since the two equilibrium conditions can be taken care of by the two sets of appropriately dimensioned stiffeners [36]. Unfortunately, this appears to be impossible for an FRC membrane because elastic interaction between two mutually orthogonal unidirectional fiber systems in FRC's would couple the solutions for each of the two equilibrium conditions. For the same reason, Michell's theory does not seem generally applicable to FRC structures reinforced with fibers of constant diameter. These observations indicate that probably in only a few special cases reinforcements patterns can be obtained which are absolutely optimal.

The use of numerical methods, particularly non-linear programming methods, complicates matters considerably. Therefore, it appears to be expedient, at least initially, to investigate the optimization problem analytically rather than numerically.

An unsuccessful attempt was made to find ways to apply the results of [17] to FRC's. The elastic strength concept as mentioned in [17] requires the structure to be statically determinate. This restriction severely limits the applicability of any optimality condition derived from [17]. Therefore, this approach has not been pursued. Moreover, completely satisfactory strength theories for composites have not yet been established. Any optimization theory based on a strength theory could be expected to be plagued by such nuisances as non-invariancy with respect to coordinate transformation or ambiguity of principal directions.

Numerical methods are virtually inevitable when a structure is to be subjected to many different loading conditions. Such a case, however, will be considered to be beyond the scope of this report, which is primarily meant as an attempt to grasp the fundamentals of reinforcement optimization.

8. Optimization of Fiber Patterns in FRC Membranes

In this section, after some preliminary remarks, an optimality condition is derived for FRC membranes. The derivation is a direct adaptation of the one given by Huang [20] for homogeneous isotropic materials. The term membrane refers to structures with in-plane stress resultants only; it implies that coupling effects are not to occur. Hence, only balanced laminates are considered.

In the third paragraph of this section a few comments are made on an illustrative example of seemingly trivial form: a wedge in tension. Details of the analysis of this wedge appear in the Appendix.

8.1. Preliminaries

a. From experiments as well as from theory it is evident that in unidirectionally reinforced FRC's the direction of maximum stiffness coincides with the fiber direction. Hence, if the FRC is required to provide a certain stiffness in a specific direction, the fiber material is most efficiently utilized when the fibers coincide with that direction [35].

b. Directional reinforcement does not affect principal stress or strain directions as long as the directions of the reinforcement coincide with the principal directions. That

is, the principal directions in an isotropic body remain unchanged when fibers are laid along those directions. As in the isotropic case, the principal stress trajectories continue to coincide with the principal strain trajectories. This follows from Hooke's law:

$$\epsilon_6 = S_{16}\sigma_1 + S_{26}\sigma_2 + S_{66}\sigma_6$$

where ϵ_6 = shear strain

σ_1, σ_2 = normal stress

σ_6 = shear stress

S_{16}, S_{26}, S_{66} = compliance coefficients

If the 1- and 2-directions are principal stress directions, then $\sigma_6 = 0$. Since the principal lines form an orthogonal system, and the fibers are laid along the principal lines, the FRC is locally specially orthotropic. Therefore, $S_{16} = S_{26} = 0$. Thus, $\epsilon_6 = 0$ regardless of the elastic constants. That is, regardless of the number of fibers if the fibers coincide with the principal stress lines. As $\epsilon_6 = 0$ is a sufficient condition for the strains to be principal strains, it follows that principal stress lines and principal strain lines, which coincide in the isotropic case, continue to coincide in the case of locally special orthotropy.

Observations a. and b. lead to the hypothesis that fiber reinforcements are optimal if the fibers follow the trajectories. And since the trajectories are not changed by the presence of the fibers, they are the same as if the material were isotropic. This is fortunate since for the determination of trajectories in isotropic bodies analytical, numerical and experimental methods are known.

8.2. Optimality Condition for FRC Membranes

An infinitesimally small element of an FRC membrane is considered. The reinforcement is treated as a specially orthotropic medium in the form of two kinds of layers. In one kind of layer the fibers coincide with the 1-direction and in the second kind the fibers coincide with the 2-direction. The 1- and 2-directions coincide with the principal stress (strain) directions. Let the two sets of layers be represented in the form of two layers. (In an actual laminate the layers of the two sets can be arranged such that no coupling effects occur.) The elastic constants for a layer are:

$$C_{11}^{(k)} = C_E \, a_k^{n_1}$$

$$C_{12}^{(k)} = C_P \, a_k^{n_3}$$

$$C_{22}^{(k)} = C_E \, a_k^{n_2}$$

where $C_{ij}^{(k)}$ = stiffness coefficient for kth layer (k = 1, 2)

C_E, C_P = constants

a_k = specific number of fibers in kth layer (specific = per unit area)

n_i = exponent, to be determined theoretically or experimentally

In the following, by strain energy, the strain energy accumulated in the reinforcement is meant. The strain energy per unit area of a membrane of unit thickness is:

$$U = 1/2\,(\sigma_1 \epsilon_1 + \sigma_2 \epsilon_2)$$

where σ_i and ϵ_i are the principal stresses and strains.

$$2U = \sigma_1 \epsilon_1 + \sigma_2 \epsilon_2 =$$

(with the usual assumption of strain compatibility between the layers)

$$= 1/2\left[(C_{11}^{(1)} + C_{11}^{(2)})\,\epsilon_1^2 + 2(C_{12}^{(1)} + C_{12}^{(2)})\,\epsilon_1\,\epsilon_2 + (C_{22}^{(1)} + C_{22}^{(2)})\,\epsilon_2^2\right]$$

$$= 1/2\Big[(b_1\, e_1\, a_1^{n_1} + b_2\, e_3\, a_1^{n_3} + b_1\, e_2\, a_1^{n_2})$$

$$+ (b_1\, e_1\, a_2^{n_2} + b_2\, e_3\, a_2^{n_3} + b_1\, e_2\, a_2^{n_1})\Big] \qquad (1)$$

$$= 1/2\left[(2U_1) + (2U_2)\right]$$

where U_i = component of U, depending on a_i

$b_1 = C_E$; $b_2 = 2\,C_P$

$e_1 = \epsilon_1^2$; $e_2 = \epsilon_2^2$; $e_3 = \epsilon_1\,\epsilon_2$

Adapting Huang's [20] analysis, an optimality condition for FRC's is derived in the following. The external work W_B can be considered as a measure of stiffness. W_B can be written as [20]:

$$-W_B = \int 1/2\,(U_1 + U_2)\,dS - P_i\,\delta_i$$

summation over repeated index

where P_i = external loads

δ_i = displacement of loadpoint of P_i

dS = surface area element

Two designs, B and $\bar{B}$, are compared which are subjected to the same loading and have the same stiffness, i.e. $W_B = W_{\bar{B}}$. Thus,

$$1/2 \int \left[(b_1 e_1 a_1^{n_1} + b_2 e_3 a_1^{n_3} + b_1 e_2 a_1^{n_2}) \right.$$

$$\left. + (b_1 e_1 a_2^{n_2} + b_2 e_3 a_2^{n_3} + b_1 e_2 a_2^{n_1}) \right] dS - P_i \delta_i$$

$$= 1/2 \int \left[(b_1 \bar{e}_1 \bar{a}_1^{n_1} + b_2 \bar{e}_3 \bar{a}_1^{n_3} + b_1 \bar{e}_2 \bar{a}_1^{n_2}) \right. \tag{2}$$

$$\left. + (b_1 \bar{e}_1 \bar{a}_2^{n_2} + b_2 \bar{e}_3 \bar{a}_2^{n_3} + b_1 \bar{e}_2 \bar{a}_2^{n_1}) \right] dS - P_i \bar{\delta}_i$$

By the principle of minimum potential energy [18], the following inequality holds:

$$1/2 \int \left[b_1 \bar{e}_1 \bar{a}_1^{n_1} + \ldots + b_1 \bar{e}_2 \bar{a}_2^{n_1} \right] dS - P_i \bar{\delta}_i$$

$$\leq 1/2 \int \left[b_1 e_1 \bar{a}_1^{n_1} + \ldots + b_1 e_2 \bar{a}_2^{n_2} \right] dS - P_i \delta_i \tag{3}$$

Substitution of (2) into (3) yields,

$$1/2 \int \left[b_1 e_1 (\bar{a}_1^{n_1} - a_1^{n_1}) + b_2 e_3 (\bar{a}_1^{n_3} - a_1^{n_3}) + b_1 e_2 (\bar{a}_1^{n_2} - a_1^{n_2}) \right.$$

$$\left. + b_1 e_1 (\bar{a}_2^{n_2} - a_2^{n_2}) + b_2 e_3 (\bar{a}_2^{n_3} - a_2^{n_3}) + b_1 e_2 (\bar{a}_2^{n_1} - a_2^{n_1}) \right] dS \geq 0 \tag{4}$$

Assuming that $\bar{a}_i$ and a_i do not differ very much, i.e. assuming that δa_i is very small in,

$$\bar{a}_i = a_i + \delta a_i$$

$\bar{a}_i^n$ then can be expanded as follows:

$\bar{a}_i^n = a_i^n + n\,a_i^{n-1}\,\delta a_i$ + terms of higher orders of δa_i, which will be neglected

$$\bar{a}_i^n = a_i^n + n\,a_i^{n-1}\,\delta a_i \tag{5}$$

With (5), (4) becomes:

$$1/2 \int \Big[(b_1 e_1 n_1 a_1^{n_1 - 1}\,\delta a_1 + b_2 e_3 a_1^{n_3 - 1}\,n_3\,\delta a_1 + b_1 e_2 n_2 a_1^{n_2 - 1}\,\delta a_1)$$

$$+ (b_1 e_1 n_2 a_2^{n_2 - 1}\,\delta a_2 + b_2 e_3 a_2^{n_3 - 1}\,n_3\,\delta a_2 + b_1 e_2 n_1 a_2^{n_1 - 1}\,\delta a_2) \Big]\,dS \geq 0$$

or, with (1),

$$1/2 \int \left(\frac{\partial U_1}{\partial a_1}\,\delta a_1 + \frac{\partial U_2}{\partial a_2}\,\delta a_2 \right) dS \geq 0 \tag{6}$$

The volume of the reinforcement can be expressed as:

$V = C \int a\,dS$, where C is a positive constant and $a = a_1 + a_2$

$$\delta V = C \int \delta a\,dS$$

$$= C \int (\delta a_1 + \delta a_2)\,dS \tag{7}$$

The design B will not have a greater reinforcement weight than any neighboring design $\bar{B}$ of the same stiffness, if the reinforcement volume of B is not greater than the reinforcement volume of $\bar{B}$. Hence, a condition has to be found for which,

$$\delta V \geq 0$$

or, from (7),

$$\int (\delta a_1 + \delta a_2)\, dS \geq 0$$

Then, from (6) it follows that this is true if,

$$\frac{\partial U_1}{\partial a_1} = \frac{\partial U_2}{\partial a_2} = C_1 \tag{8}$$

where C_1 is a positive constant, the optimality constant.

From (8) it follows that:

$$U_1 = C_1 a_1 + \text{constant}$$

where the constant is zero since $U_1 = 0$ for $a_1 = 0$, as follows from (1). Similarly, $U_2 = C_1 a_2$. Furthermore,

$$2U = U_1 + U_2 = U_1 (1 + \frac{a_2}{a_1}) = U_1 \frac{a}{a_1}$$

Thus, the optimality condition becomes:

$$\frac{2U}{a} = \frac{U_1}{a_1} = \frac{U_2}{a_2} = \text{constant} \tag{9}$$

Equation (9) is a sufficient optimality condition in the sense that it leads to a reinforcement that is not heavier than any other reinforcement of the same stiffness.

To prove the necessity of the optimality condition, the external work W_B is maximized for given V. This leads to the variational expression:

$$\delta \int \left[(b_1 e_1 a_1^{n_1} + b_2 e_3 a_1^{n_3} + b_1 e_2 a_1^{n_2}) + (b_1 e_1 a_2^{n_2} + b_2 e_3 a_2^{n_3} + b_1 e_2 a_2^{n_1}) - P_i \delta_i - \lambda C (a_1 + a_2) \right] dS = 0$$

where λ is a Lagrange multiplier.

The Euler equations reduce to:

$$b_1 e_1 n_1 a_1^{n_1 - 1} + b_2 e_3 n_3 a_1^{n_3 - 1} + b_1 e_2 n_2 a_1^{n_2 - 1} - \lambda C = 0 \qquad (10a)$$

$$b_1 e_1 n_2 a_2^{n_2 - 1} + b_2 e_3 n_3 a_2^{n_3 - 1} + b_1 e_2 n_1 a_2^{n_1 - 1} - \lambda C = 0 \qquad (10b)$$

Combination of (10a) and (10b) yields (8).

The optimum is weak and relative. Weak because of the assumption made for (5), namely that δa_i be very small [21]. Relative for two reasons: firstly, because of the statical indeterminacy, as is discussed in Paragraph 7.5. and secondly, because only the optimality of the reinforcement is considered. That means optimization with respect to the total weight of fibers and matrix together.

8.3. Example

The simplest example illustrating the potential of the method is a wedge in tension (see Figure 1). The isotropic case has been discussed in [37] and the anisotropic case in [38]. Details of the analysis appear in the Appendix. Based on data for boron-epoxy, some numerical values were computed, which are represented graphically (see Figure 2, where the optimal design is designated by A and the unidirectional one by B).

In Figure 2, Weight is the weight of that part of the wedge that is bounded by $\pm \alpha$ and r_1 and r_2. The reciprocal of the strain energy in that part of the wedge is a measure of the stiffness. The specific stiffness is defined as the reciprocal of the product of the strain energy times the weight.

For increasing wedge angle, the graph for the stiffness shows the increasing superiority of the optimal design over the unidirectional design.

In the graph for the ratio of the weight of the optimal design to the weight of the unidirectional design, a distinct decrease becomes evident at approximately 10 degrees. This can be explained as being due to the fact that past that angle the cosine starts to differ substantially from 1, and decreases more rapidly. The fiber spacing, being a function of the secant, therefore increases more rapidly which in turn causes the weight to increase at a decreasing rate. And so the ratio of the weights of the optimal and the unidirectional design decreases.

The fact that the optimal design is significantly stiffer as well as lighter than the unidirectional design is illustrated by the curve for the ratio of the specific stiffnesses.

9. Fiber LAying Gadget (FLAG)

In order to check theoretical results with experimental data, test specimens have to be made in which the fibers run along trajectories predetermined by the theory. A device, called FLAG, was made to lay a single continuous wire, thread or fiber along any predetermined curve. In this section this FLAG is described. The FLAG itself is mounted on the pen carriage of an X-Y recorder, replacing the pen assembly. The X-Y

recorder is controlled by a PDP-12 digital computer through a D/A (digital to analog) converter.

With appropriate modifications the FLAG also can be used for curved surfaces of almost every type. Its range of application, therefore, is less restricted than is the case for fiber winding techniques.

9.1. Description of the FLAG

Plumber's delight, pictured in Figures 3, 4, 5 and 6, and made of leftovers and some scrap pieces of copper tubes and brass, is FLAG, an experimental Fiber LAying Gadget.

Details of its construction are shown in Figure 5. It consists of a wheel (1) pressing the fiber onto a sticky layer and simultaneously pulling the fiber (3) from the reel (2). The wheel and reel are rigidly interconnected by the tube (4) through which the fiber is guided from the reel to the wheel. This prevents the fiber from being twisted when the wheel follows a curved trajectory and rotates around a vertical axis. The wheel-reel assembly is supported by two aligned bearings (5), allowing for unrestricted rotation around the vertical axis of the aligned bearings. The bearing assembly is fixed to one end of a curved tube (6) by means of two joints (7) and (8) which have mutually perpendicular horizontal axes. Both joints are rigid, but they are adjustable for alignment purposes. The other end of the curved tube is fixed to a horizontal bit (9) that fits in the pen carriage of the X-Y recorder (see Figure 4). Like the pen assembly, the bit can rotate freely around its horizontal axis. The wheel is covered with teflon

tape to prevent it from sticking to the layer. To avoid practical complications, no provision was made for a fiber-cutting device.

9.2. Operation of the FLAG

The FLAG, mounted on the pen carriage of the X-Y recorder, is controlled by a program in the PDP-12. Coordinates for the FLAG are calculated in digital form, and by means of a D/A converter these digital data are converted into voltages which activate the X-Y recorder. The potentiometers of the recorder provide fine adjustment, necessary to scale the computer output to the X and Y ranges of travel of the carriage.

The trajectory of the wheel is the fiber trajectory determined through theory. The trajectory of the FLAG is calculated as follows: Let b be the distance between the vertical axis of the two bearings and the point of contact between wheel and layer (see Figure 6); Let (x_1, y_1) be a point on the fiber trajectory and (x_2, y_2) the corresponding point on the trajectory of the vertical axis. Then,

$$x_2 = x_1 + b \cos \eta$$

$$y_2 = y_1 + b \cos \eta$$

where η is defined by $\text{tg}\ \eta = \frac{dy_1}{dx_1}$.

Since the fiber trajectory is known, η can be calculated for any point (x_1, y_1) and hence (x_2, y_2) can be determined. This computation is executed in the PDP-12.

9.3. Initial Results

In order to determine experimentally the effect of the interspace between the fibers on the elastic constants of the composite, several unidirectional test specimens have to be made, with different fiber spacing. To this end a program for the PDP-12 was written to make the FLAG lay fibers along straight parallel lines. The distance between the lines, i.e. the interspace between the fibers, can be adjusted by the corresponding potentiometer on the X-Y recorder.

First trial runs with thread, nylon and steel wire indicated that the fiber needed to be guided almost to the point of contact between wheel and layer. This was accomplished by flattening the lower end of the fiber guiding tube and making it follow the contour of the wheel (see Figure 7). Two different wheels of different diameter were used--the bigger one for the stiff wire and the small one for the flexible thread and nylon.

In the first prototype of the FLAG, the distance between the vertical axis and the point of contact between wheel and layer was about twice as long as it is in the current version. This was done to avoid wheel shimmy. Consequently, however, the radii of the return loops at the end of each trajectory had to be quite large. Therefore, the distance was reduced rather arbitrarily to its present value. Still no wheel shimmy occurs.

For the development of the FLAG, nylon fiber has been used almost exclusively because of easy handling combined with moderate stiffness. At first, masking tape served as the sticky layer, but the fiber usually sank into the adhesive layer of the tape,

too deeply to be covered sufficiently by the resin applied afterwards. Evaluation of other materials finally led to Booklon, a product of Boekelo Plastics N. V. in the Netherlands. Normally used to cover books, maps, etc., Booklon is a transparent, self-adhesive, plastic sheet with a protective covering which is easily peeled off. The adhesive layer is very thin and the fibers therefore are exposed sufficiently to allow satisfactory wetting during subsequent application of resin. The protective covering performs well when used as a releasing top layer. Once the resin is cured, the Booklon base layer and the protective covering can be peeled off easily. The resin content is well controlled when some pressure is applied during curing, since the fibers act as spacers. For void control, the methods commonly employed in hand lay-up techniques are used.

10. Conclusion

In the foregoing a method has been discussed to obtain relatively optimal reinforcements in FRC membranes. It is based on the observations that:

1. In FRC's the direction of maximum stiffness coincides with the fiber direction, and
2. Principal trajectories are independent of fiber reinforcement as long as the fibers coincide with the trajectories.

An optimality condition is derived which states that the specific strain energy divided by the specific number of fibers is a constant. The method is illustrated by the example of a single-layer wedge in tension. The results obtained for the example indicate the potential of the method proposed: for a 20-degree wedge, the improvement of the specific stiffness can be in the order of 50 percent. Therefore, it will be worthwhile

to investigate more complicated cases and to formulate criteria concerning the necessity of a locally orthogonal reinforcement pattern or the sufficiency of a locally unidirectional pattern. It will also be worthwhile to try to extend the optimality criterion by including the matrix in the optimization.

To take full advantage of FRC's, coupling effects have to be included because of their possibly beneficial consequences (e.g. with regard to stability).

These points illustrate the investigation of optimality of FRC plates is pertinent to both the theory and the application of FRC's.

11. Acknowledgment

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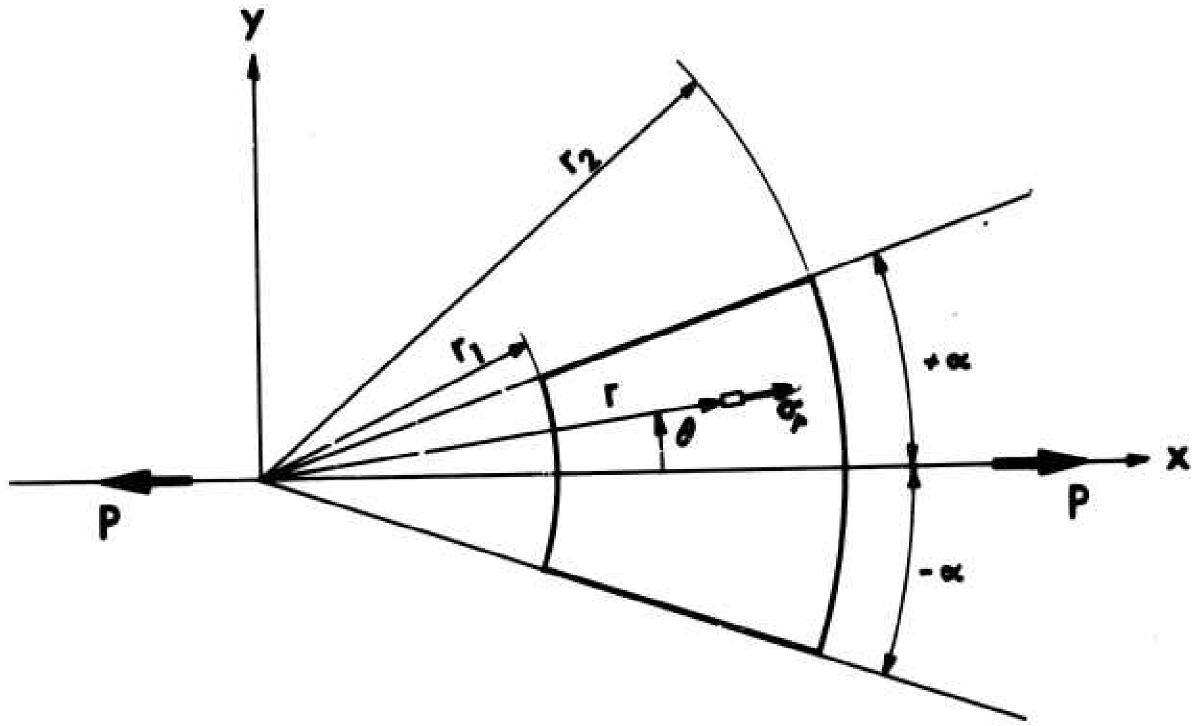


Figure 1. Coordinates and loading of wedge.

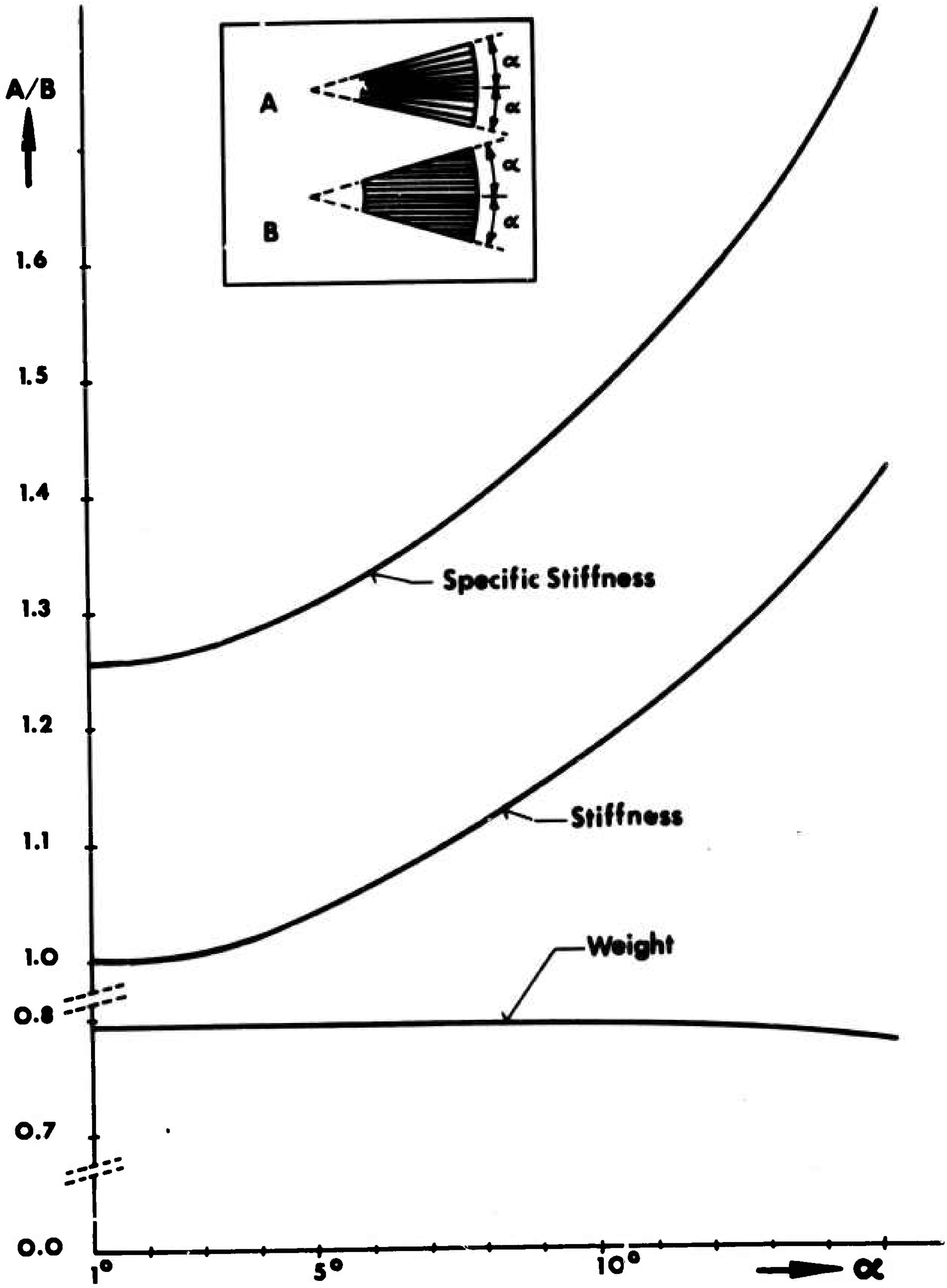


Figure 2. Ratios of Specific Stiffnesses, Stiffnesses and Weights.

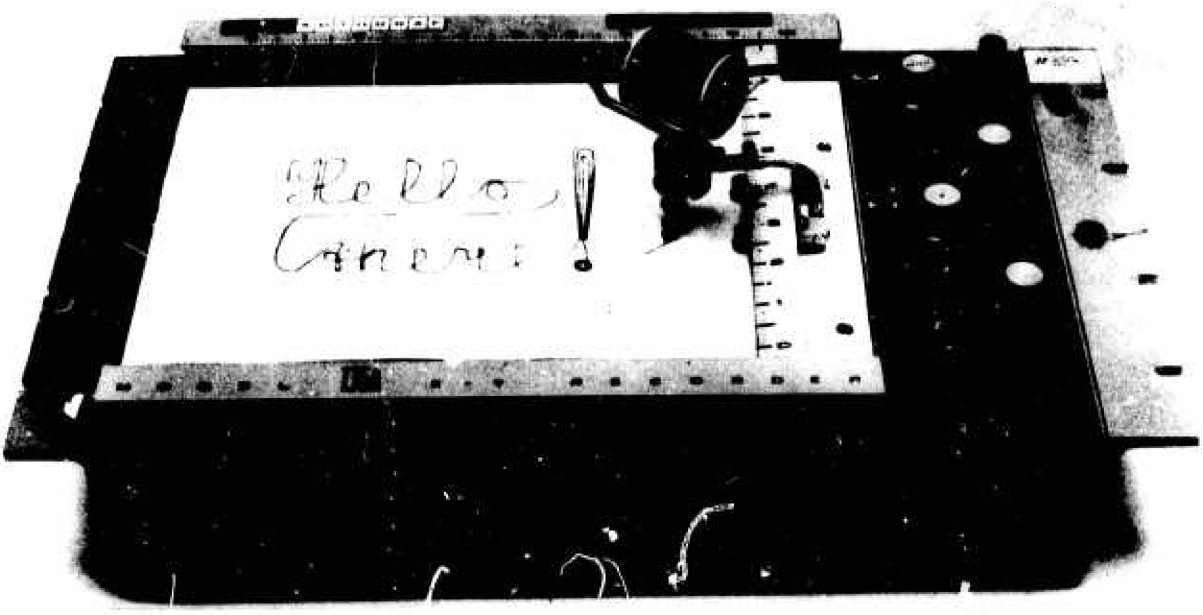


Figure 3. Fiber laying device: FLAG mounted on X–Y Recorder.

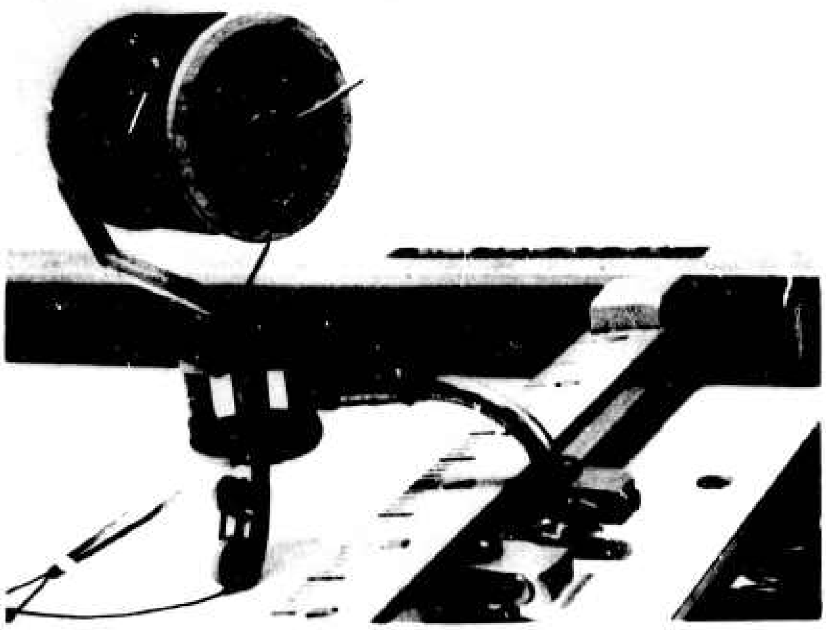


Figure 4. FLAG replacing pen-assembly.

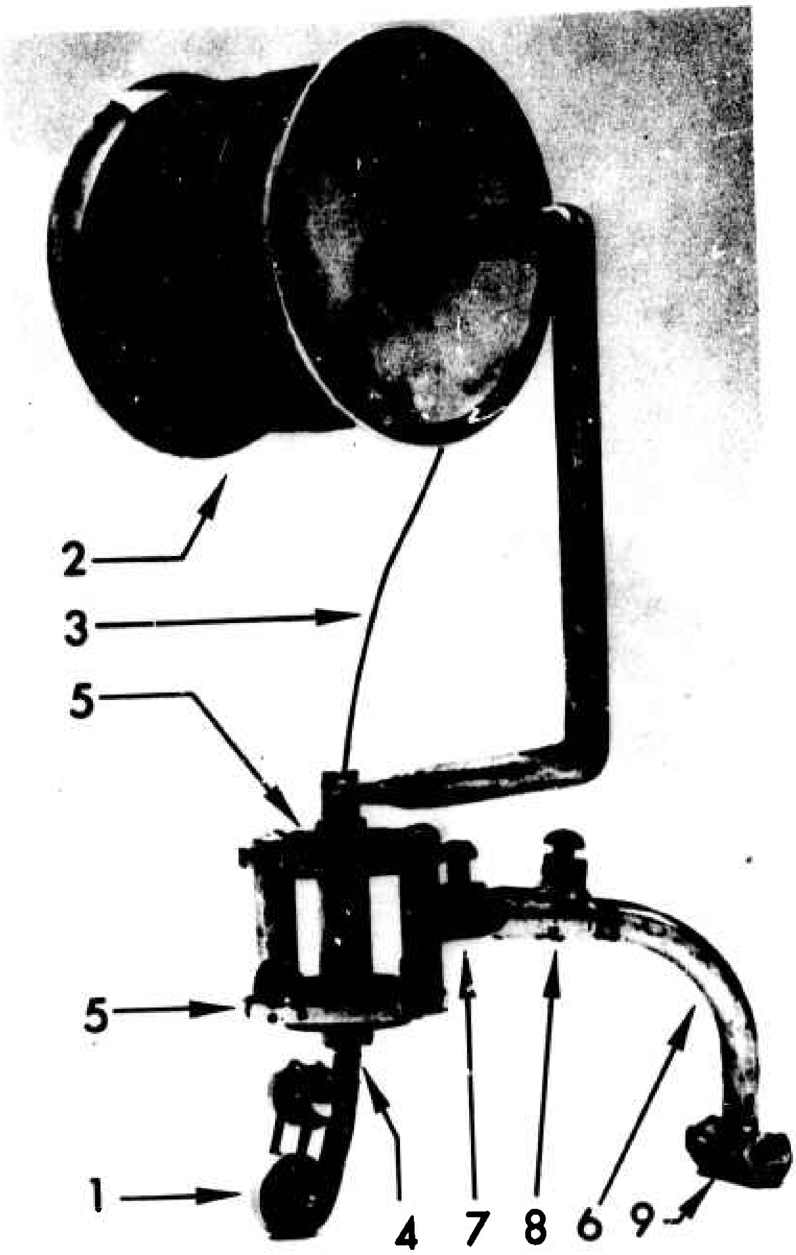
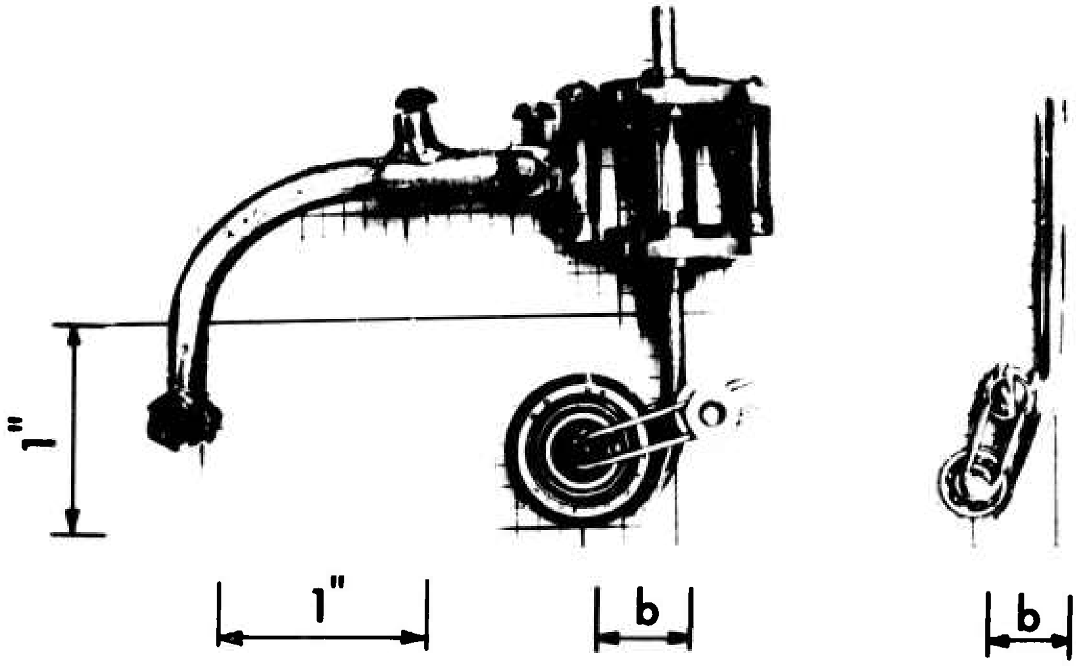


Figure 5. FLAG (see paragraph 9.1.).



(b: see paragraph 9.2.)

Figure 6. FLAG: dimensions.

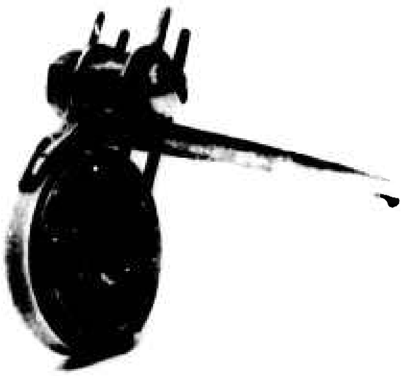


Figure 7. FLAG-wheel.
(see paragraph 9.3.)

Appendix: Analysis of the Wedge

Solutions for an isotropic wedge and an anisotropic wedge, respectively, can be found in [37], p. 97 and in [38], p. 87, respectively.

The composite is assumed to be an epoxy matrix reinforced with boron fibers. The following data are used:

$$E_f = 4.5 \times 10^6 \text{ kg/cm}^2$$

$$E_m = 0.035 \times 10^6 \text{ kg/cm}^2$$

$$\rho_f = 2.35 \text{ g/cm}^3$$

$$\rho_m = 1.15 \text{ g/cm}^3$$

As for the geometry of the wedge (see figure 1) $r_2 / r_1 = 3$ and $0 < \alpha \leq 15^\circ$

The optimal solution is compared with the solution for a wedge made of unidirectional tape, oriented in the x - direction, with the following constants:

fiber fraction: 65% by volume

$$E_x = 2.8 \times 10^6 \text{ kg/cm}^2$$

$$E_y = 0.28 \times 10^6 \text{ kg/cm}^2$$

$$V_{xy} = 0.25$$

$$G = 0.11 \times 10^6 \text{ kg/cm}^2$$

The stress distribution in an isotropic wedge is given by [37] :

$$\sigma = \frac{P \cos \theta}{r(\alpha + 1/2 \sin 2\alpha)}$$

where σ is the stress in the r-direction (figure 1): all other stress - components are zero. The wedge is assumed to be of unit thickness.

The principal trajectories are straight lines in the r - direction and concentric circles in the θ - direction.

Fibers are laid in the r - direction only.

From the optimality condition, paragraph 8.2. (g),:

$$\frac{U}{a} = C_1$$

it follows $\dfrac{\sigma^2}{E\,a} = C_1$

$$\text{i.e.} \quad Ea = \frac{C_2}{C_1} \frac{\cos^2 \theta}{r^2} \qquad \text{(A 1)}$$

where E = E - modulus in the r - direction

C_1 = constant

$$C_2 = \frac{P^2}{(\alpha + 1/2 \sin 2\alpha)^2}$$

The number of fibers is $a = a_1 \dfrac{r_1}{r} f(\theta)$ (A 2)

where a_1 = a at r_1

$f(\theta)$ = function of θ, to be determined.

For E the rule of mixtures holds, hence E is a linear function of the fiber volume fraction and consequently also a linear function of a, the number of fibers, as is shown later in this appendix.

$$E = C_3 A + C_4 \tag{A 3}$$

where $C_3 = V_{f1} \; (E_f - E_m)$

V_{f1} = fiber volume percentage for a = constant

C_4 = E_m, the E - modulus of the matrix

E_f = E - modulus of the fiber

Henceforth C_4 will be omitted since $E_m \ll E_f$ for the material considered. Then, from (A 1), (A 2) and (A 3) it follows:

$$C_3 a^2 = \frac{C_2}{C_1} \cdot \frac{\cos^2 \theta}{r^2}$$

$$\text{or} \quad f(\theta) = \sqrt{\frac{C_2}{C_1 \, C_3}} \; \frac{\cos \theta}{a_1 \; r_1} \tag{A 4}$$

Substitution of (A 4) into (A 2) gives:

$$a = a_1 \frac{r_1}{r} \sqrt{\frac{C_2}{C_1 C_3}} \quad \frac{\cos\theta}{a_1 r_1}$$

i.e. $$a = \sqrt{\frac{C_2}{C_1 C_3}} \quad \frac{\cos\theta}{r} \qquad \text{(A 5)}$$

After substitution of (A 5) into (A 3), the E - modulus becomes

$$E = \sqrt{\frac{C_2 C_3}{C_1}} \quad \frac{\cos\theta}{r} \qquad \text{(A 6)}$$

assuming $E_m \ll E_f$.

The constant C_1 can be determined from the boundary condition at $r = r_1$, $\theta = 0$. At this point the stress is maximum and therefore the number of fibers ought to be maximum. For the sake of simplicity a_1 is assumed to be 1. (If $a_1 > 1$, one or more than one additional layer is needed.)

Expression (A 5) then gives

$$1 = \sqrt{\frac{C_2}{C_1 C_3}} \quad \frac{1}{r_1}$$

i.e. $$C_1 = \frac{C_2}{C_3} \frac{1}{r_1^{\,2}} \qquad \text{(A 7)}$$

The strain follows from $\varepsilon = \frac{\sigma}{E}$, thus:

$$\epsilon = \sqrt{C_2} \quad \frac{\cos\theta}{r} \quad \sqrt{\frac{C_1}{C_2\, C_3}} \quad \frac{r}{\cos\theta}$$

or: $$\epsilon = \sqrt{\frac{C_1}{C_3}} \tag{A 8}$$

that is, the strain in the r - direction is constant

Substitute (A 7) into (A 6), respectively, to obtain

$$a = 1 \quad \frac{r_1}{r} \quad \cos\theta \tag{A 9}$$

$$E = C_3 \quad \frac{r_1}{r} \quad \cos\theta \tag{A 10}$$

(Still under the assumption that $a_1 = 1$ at $\theta = 0$).

The strain energy accumulated in the wedge between $r = r_1$ and $r = r_2$, is:

$$U_A = \iint U \, d\, S =$$

$$= C_1 \int_{r_1}^{r_2} \int_{-\alpha}^{+\alpha} a \;\; r dr \;\; d\theta$$

Substitution of (A 7) and (A 9) yields:

$$U_A = \frac{C_2}{C_3} \quad \frac{r_2 - r_1}{r_1} \quad 2 \sin\alpha \tag{A 11}$$

The relationship between fiber fraction and number of fibers is found as follows:

Let d be the fiber diameter and X_f the distance between the centers of two neighboring fibers. Then $a = d / X_f$, since a is defined as the number of fibers per unit area of the membrane (the unit area can conveniently be chosen as being d x d).

The fiber fraction (volume percentage) per unit area is:

$$V_f = \frac{\pi}{4} \frac{d^2}{X_f d}$$

or $V_f = 0.785\ a$

The total weight of the fibers is

$$W_f = \iint \rho_f V_f \ r dr \ d\theta =$$

$$= 1.57\ \rho_f\ r_1\ (r_2 - r_1)\ \sin\alpha \qquad \text{(A 12)}$$

where ρ_f = specific gravity of the fiber material

The total weight of the matrix is

$$W_m = \iint \rho_m\ (1 - V_f)\ r dr\ d\theta =$$

$$= \rho_m\ (r_2 - r_1)\ [\ (r_2 + r_1)\alpha\ - 1.57\ r_1 \sin\alpha] \qquad \text{(A 13)}$$

where ρ_m = specific gravity of the matrix material

(A 13) added to (A 12) yields the total weight (per unit thickness d) of the wedge between $r = r_1$ and $r = r_2$

$$W_A = W_f + W_m =$$

$$= (r_2 - r_1)\ [1.57\ (\rho_f - \rho_m)\ r_1\ \sin\alpha + \rho_m (r_2 + r_1)\alpha] \qquad \text{(A 14)}$$

For a wedge of anisotropic material a solution has been given in [38].

The stress distribution is given by:

$$\sigma = \frac{1}{r}\ \frac{(A \cos\theta + B \sin\theta)}{L(\theta)} \qquad \text{(A 15)}$$

where σ = stress in the r - direction; all other stress components are zero

$1 / L(\theta)$ = E - modulus in the r - direction

A, B = constants to be determined from the equilibrium conditions in

x - and y- direction:

$$\int_{-\alpha}^{+\alpha} \sigma_x\, r\, d\theta = P \qquad \text{(A 16)}$$

$$\int_{-\alpha}^{+\alpha} \sigma_y\, r\, d\theta = 0 \qquad \text{(A 17)}$$

where $\sigma_x\, rd\theta = \sigma \cos\theta\ rd\theta$

$\sigma_y\, rd\theta = \sigma \sin\theta\ rd\theta$

(A 16) and (A 17) are expanded:

$$A \int_{-\alpha}^{+\alpha} \frac{\cos^2\theta}{L(\theta)}\, d\theta + B \int_{-\alpha}^{+\alpha} \frac{\sin^2\theta \cos\theta}{L(\theta)}\, d\theta = P \qquad \text{(A 18)}$$

$$A \int_{-\alpha}^{+\alpha} \frac{\sin\theta \cos\theta}{L(\theta)} d\theta + B \int_{-\alpha}^{+\alpha} \frac{\sin^2\theta}{L(\theta)} d\theta = 0 \qquad \text{(A 19)}$$

$L(\theta)$ can be computed from the principal constants and the usual transformation:

$$L(\theta) = \frac{1}{E} = \frac{\cos^4\theta}{E_x} + \left(\frac{1}{G} - \frac{2V_{xy}}{E_x}\right) \sin^2\theta \cos^2\theta + \frac{\sin^4\theta}{E_y}$$

By numerical integration for a set of α values ranging from 1° up till and including 15° the corresponding values for A were computed. Because of the symmetry of the geometry with respects to $\theta = 0$, and since $L(\theta)$ is an even function of θ while $\sin\theta$ is not, the second integral in (A 18) and the first one in (A 19) are zero, and hence $B = 0$.

Now, an expression for the strain energy is derived:

$$U_B = \iint \sigma \epsilon \, r dr \, d\theta =$$

$$= \int_{r_1}^{r_2} \int_{-\alpha}^{\alpha} \frac{dr}{r} E \, [A \cos\theta + B \sin\theta]^2 \, d\theta =$$

$$= \int_{r_1}^{r_2} \{ A \, [A \int_{-\alpha}^{\alpha} E \cos^2\theta d\theta + B \int_{-\alpha}^{\alpha} E \sin\theta \cos\theta d\theta] +$$

$$+ B \, [A \int_{-\alpha}^{\alpha} E \sin\theta \cos\theta d\theta + B \int_{-\alpha}^{\alpha} E \sin^2\theta \, d\theta] \} \frac{dr}{r}$$

Substitution of (A18) and (A19) yields:

$$U_B = \ln\left(\frac{r_2}{r_1}\right) AP \qquad \text{(A 20)}$$

The total weight of the anisotropic wedge (per unit thickness d) between $r = r_1$

and $r = r_2$ is:

$$W_B = \frac{\alpha}{57.3} \left(V_f \rho_f + V_m \cdot \rho_m \right) \left(r_2^{\,2} - r_1^{\,2} \right) \tag{A 21}$$

(α in degrees)

Substitution of the data for the geometry and for boron-epoxy unidirectional tape as given at the beginning of this appendix, yields:

$$W_B = \frac{\alpha}{57.3} \quad (0.65 \times 2.35 + 0.35 \times 1.15) \; 8\, r_1^{\,2} \times 10^{-3} =$$

$$= 0.27 \times 10^{-3} \times r_1^{\,2} \times \alpha \quad \text{(kg)} \tag{A 22}$$

Since in the point ($r = r_1$, $\theta = 0$) the stress is maximum in both wedges, with the maximum having the same value in both cases, the results derived for the optimal solution have to be adjusted. The value of V_f for a = 1 has to be 0.65 instead of 0.785. Instead of (A 14), the following expression is used for the calculations:

$$W_A = (r_2 - r_1) \;\; [1.3\, (\rho_f - \rho_m) \;\; r_1 \sin\alpha + \rho_m \;\; (r_2 + r_1)\alpha\,] \tag{A 23}$$

Substitution of the data for the geometry and for boron and epoxy gives:

$$W_A = r_1^{\,2} \;\; 2[1.3\,(2.35 - 1.15) \sin\alpha + \; 1.15 \times 4 \times \frac{\alpha}{57.3}\,] \times 10^{-3} =$$

$$= \;[3.12 \sin\alpha + \; 0.16\alpha\,] \;\times\; r_1^{\,2} \times \; 10^{-3} \quad \text{(kg)} \tag{A 24}$$

(α in degrees)

The graphs in figure 2 were calculated using (A 11), (A 20), (A 22) and (A 24)

Security Classification

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13. ABSTRACT

A method is proposed to obtain relatively optimal reinforcements in fiber-reinforced composite membrane. The method is based on the fact that: a) A reinforcing fiber is most efficiently utilized when it coincides with the direction of maximum required stiffness. b) The directions of principal trajectories are not dependent on any fiber reinforcement as long as the fibers coincide with those directions.

An optimality condition for the fiber reinforcement is derived. The derivation is an adaptation of one given in the literature for isotropic homogeneous materials. The optimality condition derived states that the specific strain energy divided by the specific number of fibers is a constant, for maximum stiffness at a given weight, and minimum weight for given stiffness. Since optimization of the reinforcement only is considered and optimization of the matrix is not included, the resulting optimum is relative.

Mathematically, the optimum is weak. As an example, some numerical values for a boron-epoxy wedge in tension were computed.

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